The size of vector spaces

Our next goal is to understand the "size" of a subspace (is it a line, plane, etc?). To make this notion precise, we need a few important concepts

Linear Independance

Def: A set of vectors $v_1, \dots, v_p \in \mathbb{R}^n$ is **linearly independent** if $x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0 \implies x_1 = x_2 = \cdots = x_n = 0$

(ie. the only linear combination of $v_1, \dots, v_p$ equal to 0 is $0v_1 + 0v_2 + \cdots + 0v_p$)

Otherwise, the vectors are **linearly dependent**

Ex:
- $\{\vec{0}\}$ is linearly dependent since $2 \cdot \vec{0} = \vec{0}$
- If $v \neq \vec{0}$, then $\{v\}$ is linearly independent since if $xv = \vec{0}$, then $x = 0$.

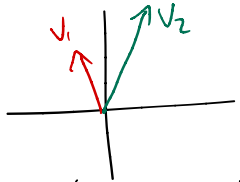
Ex: Two vectors $v_1, v_2 \in \mathbb{R}^n$ are linearly independent if and only if they are not multiples of each other

- $v_1 = cv_2 \implies 1 \cdot v_1 - cv_2 = 0 \implies v_1$ and v_2 are linearly dep (similar if $v_2 = cv_1$)
- v_1, v_2 L.I. $\implies$ there are c_1, c_2 not both 0 such that $c_1 v_1 + c_2 v_2 = 0$

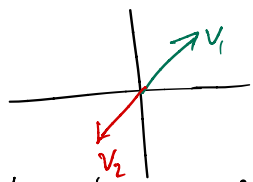
$\Rightarrow$ if $c_1 \neq 0$, then $v_1 = -\frac{c_2}{c_1} v_2$
if $c_2 \neq 0$, then $v_2 = -\frac{c_1}{c_2} v_1$

(i.e. v_1 is a mult. of v_2 or v_2 is a mult. of v_1)

Geometrically:



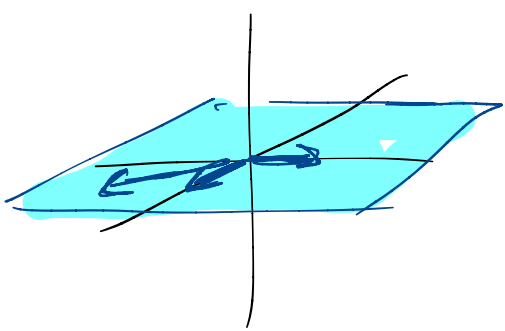
Linearly independent



Linearly dependent

More generally, $\{v_1, \ldots, v_p\}$ is linearly dependent if and only if (at least) one of the vectors is a linear combination of the others

Ex: The vectors $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\-3\\0\end{bmatrix}$ lie in the xy-plane in $\mathbb{R}^3$



Notice:

$$\begin{bmatrix}2\\-3\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\\0\end{bmatrix}$$

We can rewrite this as:

$$2\begin{bmatrix}1\\0\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\\0\end{bmatrix} - 1\begin{bmatrix}2\\-3\\0\end{bmatrix} = \vec{0}$$

Hence $x_1\begin{bmatrix}1\\0\\0\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}2\\-3\\0\end{bmatrix} = \vec{0} \not\Rightarrow x_1 = x_2 = x_3 = 0$

and so they are linearly dependent.

(You will prove this in general on next worksheet)

There is a systematic way to determine whether vectors are independent.

We can reframe linear independence as a question about $N(A)$:

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

$$\underbrace{[v_1\ v_2 \cdots v_p]}_{A} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$Ax = 0$$

So $v_1, \ldots, v_p$ are linearly independent if and only if $N(A) = \{0\}$, where $A = [v_1 \cdots v_p]$.

When is $N(A) = \{0\}$? When $Ax = 0$ has no free variables

So $v_1, \ldots, v_p$ are linearly independent $\iff$ A has a pivot in every column

Ex: Determine whether $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} \right\}$ is linearly independent.

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ -1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix} \xrightarrow{\text{reduce}} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -26 \end{bmatrix}$$

A pivot in each column $\Rightarrow$ L.I

Basis

Def: Let V be a subspace of $\mathbb{R}^n$

An ordered set of vectors $B = \{b_1, \dots, b_k\}$ is a basis for V if:

a) $\{b_1, \dots, b_k\}$ is linearly independent

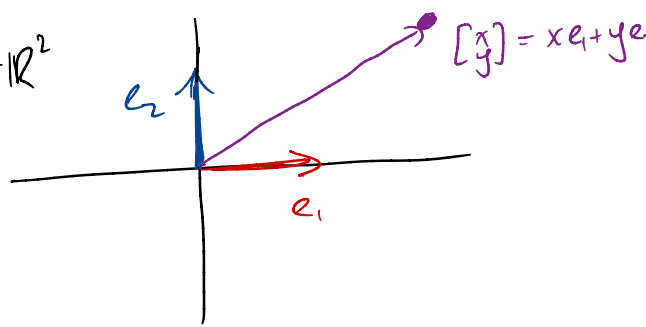
b) $\text{Span}\{v_1, \dots, v_k\} = V$. (i.e. every $v \in V$ is a L.C. of $v_1, \dots, v_k$)

Ex: $\mathcal{E} = \{\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{e_1}, \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{e_2}\}$ is a basis for $\mathbb{R}^2$ (standard basis)

a) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ has a pivot in each column $\Rightarrow \{e_1, e_2\}$ L.I.

b) Any $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$ can be written as a L.C. of $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$. That is, $\begin{bmatrix}x\\y\end{bmatrix} = x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}$

Hence $\text{Span}\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\} = \mathbb{R}^2$



More generally,

$\mathcal{E} = \{e_1, e_2, \dots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

This is not the only basis of $\mathbb{R}^n$. There are many:

Ex: Let A be an invertible $n \times n$ matrix. Then the columns of A form a basis of $\mathbb{R}^n$.

a) Since A is invertible, it has a pivot in every column $\Rightarrow$ L.I.

b) Since A is invertible, $Ax = b$ has a solution $\forall\, b \in \mathbb{R}^n \Rightarrow b \in C(A) \Rightarrow C(A) = \mathbb{R}^n$

E.g. $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$ is invertible (check that there are 2 pivots)

Hence $\left\{ \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$